

ON THE CONVERGENCE OF EIGENVALUES AND EIGENFUNCTIONS OF THE LAPLACIAN WITH WENTZELL-ROBIN BOUNDARY CONDITION

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ABSTRACT

In this paper we are interested in the problem of convergence of the eigenvalues and eigenfunctions of the Laplacian with Wentzell-Robin boundary condition to the eigenvalues and eigenfunctions of the Laplacian with Dirichlet boundary condition when the Robin parameter tends to infinity. We show in particular that the convergence of eigenfunctions is better than the usual internal convergence in $H^1(\Omega)$ and boundary convergence in $L^2(\Gamma)$.

KEYWORDS: Laplacien, Went Zell-Robin Boundary Conditions

Mathematics Subject Classification: 35P15, 35J25

1. INTRODUCTION

Let Ω be a sufficiently smooth domain (at least C^2) and let $\Gamma_D = \cup_{1 \leq j \leq N_D} \Gamma_{D,j}$, $\Gamma_{Ne} = \cup_{1 \leq j \leq N_N} \Gamma_{Ne,j}$, $\Gamma_R = \cup_{1 \leq j \leq N_R} \Gamma_{R,j}$ and $\Gamma_{WR} = \cup_{1 \leq j \leq N_{WR}} \Gamma_{WR,j}$ respectively denote regions where

Dirichlet BC, Neumann BC, Robin BC and Wentzell-Robin BC are imposed.

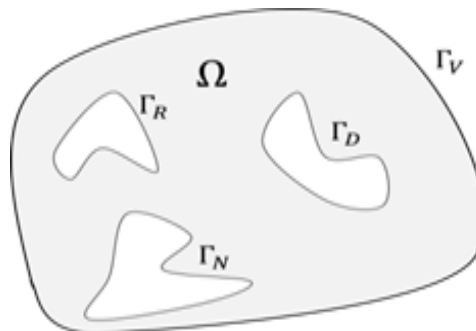


Figure 1: An Example of an Open Domain with Different Conditions on Boundary

$$u = 0 \text{ on } \Gamma_D \quad (1)$$

$$\frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_N \quad (2)$$

$$\beta_j u + \frac{\hbar^2}{2m} \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_j; 1 \leq j \leq N_R \quad (3)$$

$$\alpha_j u + \frac{\partial u}{\partial n} - \delta_j \Delta_\partial u = 0 \text{ sur } \Gamma_j; 1 \leq j \leq N_{WR} \quad (4)$$

The question we want to answer in this article is:

« For a Fixed Domain

How Do Eigenvalues and Eigen Functions of the Wentzell-Robin - Laplacian Depend on the Parameters of the Boundary Conditions ? »

The Robin boundary condition (also known as Fourier boundary condition or of the third kind) and Wentzell-Robin boundary condition are called artificial conditions and play important roles in many physical phenomena governed by partial differential equations. We can cite as examples: thin layers problems, electromagnetic, acoustics, vibration, heat problems.etc. Wentzell condition, which was introduced in [1], can be understood as an additional potential energy from the boundary $\delta \int_{\partial\Omega} |\nabla_{\partial\Omega} u|^2$ which adds to the internal potential energy through the outflow. In the dynamic case it expresses an energy supplement of both potential and kinetic kind : $\frac{1}{2} (\int_{\partial\Omega} |\partial_t u|^2 + \delta \int_{\partial\Omega} |\nabla_{\partial\Omega} u|^2)$. The parameters α and β appearing in the Wentzell-Robin condition are either operators or simply constants imposed by the physical nature and/or by the geometric nature of the problem considered.

There is a large literature on the subject, we cannot make a complete list in this paper, but for some aspects of the physical interpretation as well as mathematical and numerical study of such conditions the following articles and their references can be consulted: [2], [3],[4],[5],[6], [7], [8], [9].

2. PRELIMINARIES AND NOTATIONS

Let us consider the Dirichlet eigen value problem of the Laplacian in a C^2 -bounded domain $\Omega \subset \mathbb{R}^2$:

$$-\Delta u = \lambda u \text{ in } \Omega \quad (5)$$

$$u = 0 \text{ on } \partial\Omega \quad (6)$$

There exist a sequence $(\lambda_k, u_k)_{k=1,\infty}$ solutions of the system and such that:

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \dots < \lambda_k \leq \dots < \dots \quad (7)$$

Counted with their multiplicities, and $(u_k)_{k=1,\infty}$ is orthonormal in $L^2(\Omega)$ satisfying:

$$\lambda_k = \max_{A \in H_0^1(\Omega)} \min_{v \in A, v \neq 0, \text{codim}(A)=k-1} R(\Delta, D, v) \quad (8)$$

Where $R(\Delta, D, v)$ denotes the classical Rayleigh quotient i.e. :

$$R(\Delta, D, v) = \frac{\int_{\Omega} |\nabla v|^2}{\int_{\Omega} |v|^2} \quad (9)$$

Consider, on the other, the following Wentzell-Robin eigenvalues problem:

$$-\Delta u = \lambda u \text{ in } \Omega \quad (10)$$

$$\alpha u + \frac{\partial u}{\partial n} - \delta \Delta_{\partial} u = 0 \text{ on } \partial\Omega \quad (11)$$

There exist a sequence $(\lambda_k(\alpha, \delta), u_k(\alpha, \delta))_{k=1,\infty}$ solutions of the system and such that:

$$0 < \lambda_1(\alpha, \delta) < \lambda_2(\alpha, \delta) \leq \lambda_3(\alpha, \delta) \dots < \lambda_k(\alpha, \delta) \leq \dots < \dots \quad (12)$$

And we can again make the family $(u_k(\alpha, \delta))_{k=1,\infty}$ orthonormal in $L^2(\Omega)$, and such that the eigenvalues satisfy:

$$\lambda_k(\alpha, \delta) = \max_{A \subset \mathcal{H}(\Omega)} \min_{v \in A, v \neq 0, \text{codim}(A)=k-1} R(\Delta, \alpha, \delta, v) \tag{13}$$

Where

$$\mathcal{H}(\Omega) = \left\{ v \in H^1(\Omega); u|_{\Gamma_i} \in H_0^1(\Gamma_i) \forall 1 \leq i \leq N_V; u|_{\Gamma_j} = 0, 1 \leq j \leq N_D \right\} \tag{14}$$

and $R(\Delta, \alpha, \delta, v)$ denotes the corresponding Rayleigh quotient i.e. :

$$R(\Delta, \alpha, \delta, v) = \frac{\int_{\Omega} |\nabla v|^2 + \alpha \int_{\Gamma_V} |v|^2 + \delta \int_{\Gamma_V} |\nabla_{\Gamma} v|^2}{\int_{\Omega} |v|^2} \tag{15}$$

We give in **Section 3** a an answer to the question posed in the introduction which can be seen both qualitative and quantitative. In **Section 4** we expose the proof of the main result in several steps. Finally we conclude, in **section 5**, this paper with some remarks, perspectives and some open problems.

3. MAIN RESULTS

We give below the first result concerning the dependence, on parameters α and δ , of the eigenvalues and eigenfunctions of the Laplacian with Went zell-Robin boundary condition and the main result of convergence of the eigenfunctions when Robin parameter tends to infinity:

Theorem 1

(a) $\forall k \geq 1$, the Went zell-Robin eigenvalue $\lambda_k(\alpha, \delta)$ is continuous converge to $\lambda_k(D)$ when $\alpha \rightarrow +\infty$ and satisfy :

$$\forall \alpha_1 \geq \alpha_2; \lambda_k(\alpha_1, \delta) \geq \lambda_k(\alpha_2, \delta) \tag{16}$$

$$\forall \delta_1 \geq \delta_2; \lambda_k(\alpha, \delta_1) \geq \lambda_k(\alpha, \delta_2) \tag{17}$$

$$\forall \alpha \geq 0; \lambda_k(D) \geq \lambda_k(\alpha, \delta) \tag{18}$$

(b) $\forall k \geq 1$, the Wentzell-Robin eigenfunction $u_k(\alpha, \delta)$ converge to $u_k(D)$ in the following sense :

$$u_k(\alpha, \delta) \rightarrow u_k(D) \text{ in } H^2(\Omega) \tag{19}$$

$$u_k(\alpha, \delta) \rightarrow 0 \text{ in } H^1(\Gamma_D \cup \Gamma_R \cup \Gamma_V) \tag{20}$$

and

$$\frac{\partial u_k(\alpha, \delta)}{\partial n} \rightarrow \frac{\partial u_k(D)}{\partial n} \text{ in } L^2(\Gamma_R \cup \Gamma_V) \tag{21}$$

Recall that we had already shown in [8] and [9] the convergence of eigen functions in $H^2(\Omega)$ in the case where the boundary condition is a pure Robin BC to the eigen functions of Dirichlet when the parameter α tends to infinty and we had shown the convergence to the eigen functions with Neumann BC when the parameter α tends to zero but with the restriction to the case of convex domains. In this context Theorem 1 is an extension, in the Dirichlet case, of this above result, however here we do not treat the case of convergence to the Neumann problem.

Note also that in another work being published we got finer estimates for the principal eigen value and its asymptotic expansion as a function of α .

4. PROOF OF THE MAIN RESULTS

The proof is divided into several steps:

4.1. The Eigenvalues as a Function of Parameters Went Zell-Robin

There is no difficulty in noting that

$$R(\Delta, \alpha_1, \delta, v) \leq R(\Delta, \alpha_2, \delta, v) \quad \forall \alpha_1 \geq \alpha_2 \geq 0, \forall \delta \geq 0$$

and

$$R(\Delta, \alpha, \delta_1, v) \leq R(\Delta, \alpha, \delta_2, v) \quad \forall \delta_1 \geq \delta_2 \geq 0, \forall \alpha \geq 0$$

On the other hand, for any fixed v , the Rayleigh quotient is clearly a continuous function with respect to parameters α and δ .

Of course both the continuity inequalities remain true through the introduction of the minimum :

$$\min_{v \in A, v \neq 0, \text{codim}(A)=k-1} \text{ and then the maximum : } \max_{A \subset \mathcal{H}(\Omega)}$$

On the other hand we have

$$R(\Delta, \alpha, \delta, w) = \frac{\int_{\Omega} |\nabla w|^2 + \alpha \int_{\Gamma_V} |w|^2 + \delta \int_{\Gamma_V} |\nabla_{\Gamma} w|^2}{\int_{\Omega} |w|^2} \geq$$

$$R(\Delta, w) = \frac{\int_{\Omega} |\nabla w|^2}{\int_{\Omega} |w|^2} \quad \forall w \in H_0^1(\Omega)$$

And since the space $H_0^1(\Omega)$ is included in $\mathcal{H}(\Omega)$ the inequality $\forall \alpha \geq 0; \lambda_k(D) \geq \lambda_k(\alpha, \delta)$ is immediate.

This completes the proof of the first part of theorem 1.

4.2. Convergence of Went Zell-Robin Eigen Functions

The proof is also done in several steps: We first begin by establishing identities Rellich type for eigenfunctions inspired by a technique usual multipliers control theory (**Lemma 1** and **Lemma 2**), then in the last step we analyze the passages to the limit.

Step 1

Lemma 1

For any real parameters α and δ and any eigen function $u_k(\alpha, \delta)$ we have:

$$\begin{aligned} & \int_{\Omega} \left\{ |\partial_{xx} u_k(\alpha, \delta)|^2 + |\partial_{yy} u_k(\alpha, \delta)|^2 + 2|\partial_{xy} u_k(\alpha, \delta)|^2 \right\} \\ & + \int_{\Gamma_R \cup \Gamma_V} \left\{ |\partial_r u_k(\alpha, \delta)|^2 + |\partial_n u_k(\alpha, \delta)|^2 \right\} (\tau_x \partial_r \tau_y - \tau_y \partial_r \tau_x) \\ & + 2\alpha \int_{\Gamma_R \cup \Gamma_V} |\partial_r u_k(\alpha, \delta)|^2 + 2 \int_{\Gamma_V} |\nabla_{\Gamma} \partial_r u_k(\alpha, \delta)|^2 = \\ & (\lambda_{\alpha})^2 \int_{\Omega} |u_k(\alpha, \delta)|^2 \end{aligned} \tag{22}$$

Proof of Lemma 1

We multiply the Went zel-Robin eigen value equation by $-\Delta u_k(\alpha, \delta)$ and integrate. We then obtain:

$$\int_{\Omega} |\Delta u_k(\alpha, \delta)|^2 = \lambda_k(\alpha, \delta) \int_{\Omega} |\nabla u_k(\alpha, \delta)|^2 + \delta \lambda_k(\alpha, \delta) \int_{\partial\Omega} |\nabla_{\Gamma} u_k(\alpha, \delta)|^2 + \alpha \lambda_k(\alpha, \delta) \int_{\Omega} |(u_k(\alpha, \delta))'|^2 \tag{23}$$

Moreover one has

$$|\lambda_k(\alpha, \delta)|^2 \int_{\Omega} |u_{\lambda, \alpha, \beta}|^2 = \int_{\Omega} |\Delta u_k(\alpha, \delta)|^2 \tag{24}$$

and

$$\int_{\Omega} |\Delta u_k(\alpha, \delta)|^2 = \int_{\Omega} |\partial_{xx} u_k(\alpha, \delta)|^2 + |\partial_{yy} u_k(\alpha, \delta)|^2 + 2(\partial_{xx} u_k(\alpha, \delta)) (\partial_{yy} u_k(\alpha, \delta)) \tag{25}$$

Let us develop the last term as follows :

$$\int_{\Omega} 2(\partial_{xx} u_k(\alpha, \delta)) (\partial_{yy} u_k(\alpha, \delta)) = \int_{\Omega} |\partial_{xy} u_k(\alpha, \delta)|^2 + \int_{\partial\Omega} (\partial_x u_k(\alpha, \delta) \eta_x) (\partial_{yy} u_k(\alpha, \delta)) - (\partial_x u_k(\alpha, \delta)) (\partial_{xy} u_k(\alpha, \delta) \eta_y) \tag{26}$$

Define I as the boundary integral in the above identity. To do the calculations on the boundary we use an open cover $(\Gamma_i)_{i=1, \dots, N}$ such that $\Gamma_i = \{(x, y); y = g_i(x)\}$ and where g_i represents denote local chart of class C^2 , and such that we can write locally the normal vector as $\vec{n} = (n_x, n_y) = \frac{1}{\sqrt{1+(g')^2}}(-g'(x), 1)$ and the tangential vector as

$$\vec{\tau} = (\tau_x, \tau_y) = \frac{1}{\sqrt{1+(g')^2}}(1, g'(x)).$$

Integrations by parts are, then, performed classically by straightening each part of the boundary. We then get: $I =$

$$-\int_{\partial\Omega} (\partial_{\tau} u_k(\alpha, \delta) \tau_x - \partial_n u_k(\alpha, \delta) \tau_y) \partial_{\tau} (\partial_{\tau} u_k(\alpha, \delta) \tau_y + \tau_x \partial_n u_k(\alpha, \delta)) \tag{27}$$

Let us decompose the integral in eight parts : $I_1 + I_2 + \dots + I_8$ as follows :

$$I_1 = \int_{\partial\Omega} \tau_x \tau_y \partial_{\tau} u_k(\alpha, \delta) \partial_{\tau^2} u_k(\alpha, \delta) ; I_2 = \int_{\partial\Omega} \tau_x |\partial_{\tau} u_k(\alpha, \delta)|^2 \partial_{\tau} \tau_y ;$$

$$I_3 = \int_{\partial\Omega} \tau_y^2 \partial_n u_k(\alpha, \delta) \partial_{\tau^2} u_k(\alpha, \delta) ; I_4 = \int_{\partial\Omega} \tau_y \partial_{\tau} \tau_y \partial_n u_k(\alpha, \delta) \partial_{\tau} u_k(\alpha, \delta) ;$$

$$I_5 = \int_{\partial\Omega} \tau_x^2 \partial_{\tau} u_k(\alpha, \delta) \partial_{\tau} \partial_n u_k(\alpha, \delta) ; I_6 = \int_{\partial\Omega} \tau_x \partial_{\tau} \tau_x \partial_{\tau} u_k(\alpha, \delta) \partial_n u_k(\alpha, \delta) ;$$

$$I_7 = \int_{\partial\Omega} \tau_x \partial_{\tau} \tau_x \partial_{\tau} u_k(\alpha, \delta) \partial_n u_k(\alpha, \delta) \text{ and } I_8 = \int_{\partial\Omega} \tau_y \partial_{\tau} \tau_x (\partial_n u_k(\alpha, \delta))^2$$

Then

$$I_1 = -\int_{\partial\Omega} \tau_x \tau_y \partial_{\tau} u_k(\alpha, \delta) \partial_{\tau^2} u_k(\alpha, \delta) = -\frac{1}{2} \int_{\partial\Omega} \tau_x \tau_y \partial_{\tau} |(\partial_{\tau} u_k(\alpha, \delta))|^2$$

$$= \frac{1}{2} \int_{\partial\Omega} |(\partial_\tau u_k(\alpha, \delta))|^2 (\tau_x \partial_\tau(\tau_y) + \tau_y \partial_\tau(\tau_x)) \quad (28)$$

$$I_2 = \int_{\partial\Omega} |\partial_\tau u_k(\alpha, \delta)|^2 (\tau_x \partial_\tau \tau_y) \quad (29)$$

$$I_1 + I_2 = \frac{1}{2} \int_{\partial\Omega} |\partial_\tau u_k(\alpha, \delta)|^2 (\tau_y \partial_\tau(\tau_x) - \tau_x \partial_\tau(\tau_y)) \quad (30)$$

$$I_7 + I_8 = \frac{1}{2} \int_{\partial\Omega} |\partial_n u_k(\alpha, \delta)|^2 (\tau_y \partial_\tau(\tau_x) - \tau_x \partial_\tau(\tau_y)) \quad (31)$$

$$\begin{aligned} I_3 &= \int_{\partial\Omega} \tau_y^2 \partial_n u_k(\alpha, \delta) \partial_{\tau^2} u_k(\alpha, \delta) \\ &= - \int_{\partial\Omega} \tau_y^2 \partial_\tau (\partial_n u_k(\alpha, \delta)) \partial_\tau \tau_y^2 \end{aligned} \quad (32)$$

$$I_5 = - \int_{\partial\Omega} \tau_x^2 \partial_\tau u_k(\alpha, \delta) \partial_n u_k(\alpha, \delta) \quad (33)$$

One obtain

$$\begin{aligned} I_5 + I_3 &= \\ & \int_{\partial\Omega} \partial_\tau u_k(\alpha, \delta) \partial_\tau ((\partial_n u_k(\alpha, \delta))(-\tau_x^2 - \tau_y^2)) - \int_{\partial\Omega} (\partial_n u_k(\alpha, \delta)) (\partial_\tau u_k(\alpha, \delta)) \partial_\tau (\tau_y^2) \\ &= - \int_{\partial\Omega} \partial_\tau u_k(\alpha, \delta) \partial_\tau (\partial_n u_k(\alpha, \delta)) - \int_{\partial\Omega} (\partial_n u_k(\alpha, \delta)) \partial_\tau u_k(\alpha, \delta) \partial_\tau (\tau_y^2) \end{aligned} \quad (34)$$

$$I_4 + I_6 = \frac{1}{2} \int_{\partial\Omega} \partial_\tau u_k(\alpha, \delta) (\partial_n u_k(\alpha, \delta)) (\partial_\tau \tau_y^2 - \partial_\tau \tau_x^2) \quad (35)$$

$$\begin{aligned} I_3 + I_5 + I_4 + I_6 &= \tilde{B} \\ &= - \int_{\partial\Omega} \partial_\tau u_k(\alpha, \delta) \partial_\tau (\partial_n u_k(\alpha, \delta)) - \frac{1}{2} \int_{\partial\Omega} \partial_\tau u_k(\alpha, \delta) (\partial_n u_k(\alpha, \delta)) \partial_\tau (\tau_y^2 - \tau_x^2) \\ &= - \int_{\partial\Omega} \partial_\tau u_k(\alpha, \delta) \partial_\tau (\partial_n u_k(\alpha, \delta)) \end{aligned} \quad (36)$$

$$\begin{aligned} I_1 + I_2 + I_7 + I_8 &= \\ & \frac{1}{2} \int_{\partial\Omega} ((\partial_\tau u_k(\alpha, \delta))^2 - (\partial_n u_k(\alpha, \delta))^2) (\tau_y \partial_\tau \tau_x - \tau_x \partial_\tau \tau_y) \end{aligned} \quad (37)$$

And finally we have

$$\begin{aligned} \tilde{B} &= - \int_{\partial\Omega} \partial_\tau u_k(\alpha, \delta) \partial_\tau (\partial_n u_k(\alpha, \delta)) \\ &= - \int_{\partial\Omega} \partial_\tau u_k(\alpha, \delta) \partial_\tau (\Delta_r u_k(\alpha, \delta) - \alpha u_k(\alpha, \delta)) \\ &= - \int_{\partial\Omega} \partial_\tau u_k(\alpha, \delta) \partial_\tau \Delta_r u_k(\alpha, \delta) + \alpha \int_{\partial\Omega} \partial_\tau u_k(\alpha, \delta) \partial_\tau u_k(\alpha, \delta) \\ &= - \int_{\partial\Omega} |\Delta_r \partial_\tau u_k(\alpha, \delta)|^2 + \alpha \int_{\partial\Omega} (\partial_\tau u_k(\alpha, \delta))^2 \\ &= \int_{\partial\Omega} |\Delta_r \partial_\tau u_k(\alpha, \delta)|^2 + \alpha \int_{\partial\Omega} (\partial_\tau u_k(\alpha, \delta))^2 \\ &= \int_{\partial\Omega} |\partial_{\tau^2} u_k(\alpha, \delta)|^2 + \alpha \int_{\partial\Omega} (\partial_\tau u_k(\alpha, \delta))^2 \end{aligned} \quad (38)$$

Step 2

Lemma 2: Let $(q_k)_{1 \leq k \leq n}$ be a vector field in $W_1^1(\Omega)$ such that $\sum_{1 \leq k \leq n} q_k n_k = 1$ on $\partial\Omega$, then we have the following identities :

$$\begin{aligned} & \frac{1}{2} \int_{\Gamma_R \cup \Gamma_V} \{ |\partial_\tau u_k(\alpha, \delta)|^2 - |\partial_n u_k(\alpha, \delta)|^2 \} = \\ & \frac{1}{2} \int_\Omega |\nabla u_k(\alpha, \delta)|^2 \frac{\partial q_i}{\partial x_i} - \int_\Omega \frac{\partial u_k(\alpha, \delta)}{\partial x_l} \frac{\partial q_i}{\partial x_l} \frac{\partial u_k(\alpha, \delta)}{\partial x_i} \\ & - \frac{\lambda_k(\alpha, \delta)}{2} \int_\Omega \left| (u_k(\alpha, \delta))^2 \right| \frac{\partial q_i}{\partial x_i} + \frac{\lambda_k(\alpha, \delta)}{2} \int_{\partial\Omega} \left| (u_k(\alpha, \delta))^2 \right| \end{aligned} \tag{39}$$

and

$$\begin{aligned} & - \frac{1}{2} \int_{\Gamma_R \cup \Gamma_V} |\partial_n u_k(D)|^2 = \\ & \frac{1}{2} \int_\Omega |\nabla u_k(D)|^2 \frac{\partial q_i}{\partial x_i} - \int_\Omega \frac{\partial u_k(D)}{\partial x_l} \frac{\partial q_i}{\partial x_l} \frac{\partial u_k(D)}{\partial x_i} - \frac{\lambda_k(D)}{2} \int_\Omega \left| (u_k(D))^2 \right| \frac{\partial q_i}{\partial x_i} \end{aligned} \tag{40}$$

Sketch of the Proof of Lemma 2

The identities of Lemma 2 are easily obtained by multiplying successively the kth eigenvalues Laplace-Went zell-Robin problem by $u_k(\alpha, \delta)$ and the kth eigenvalues Laplace-Dirichlet problem by $u_k(D)$ and integrating by parts.

Step 3

Note first that there is no particular difficulty to show for every fixed and positive δ that the increasing and bounded sequence $\lambda_k(\alpha, \delta)$ converges to $\lambda_k(D)$. The fact that the sequence $u_k(\alpha, \delta)$ is bounded in $H^1(\Omega)$ then deduce immediately. So it is easy to identify the limit and to prove that $u_k(\alpha, \delta)$ converges in $H^1(\Omega)$ to $u_k(D)$ when α tends to infinity. To be convinced we can use, for example, the following identity :

$$\begin{aligned} & \int_\Omega |\nabla (u_k(\alpha, \delta) - u_k(D))|^2 = \lambda_k(\alpha, \delta) \int_\Omega |u_k(\alpha, \delta) - u_k(D)|^2 + \\ & (\alpha_k(\alpha, \delta) - \alpha_k(D)) \int_\Omega |u_k(D)|^2 - \int_{\partial\Omega} u_k(\alpha, \delta) \frac{\partial u_k(D)}{\partial n} \end{aligned} \tag{41}$$

Now observe that in the case where the quantity $(\tau_x \partial_\tau \tau_y - \tau_y \partial_\tau \tau_x)$ is positive then the convergence $u_k(\alpha, \delta) \rightarrow 0$ in $H^1(\partial\Omega)$ is immediate. Note that this quantity is positive especially when the domain is convex [9].

In the general case, we know from the first identity of Lemma 2 that there exists a constant C independent of the parameters of the problem and such that:

$$\int_{\Gamma_R \cup \Gamma_V} |\partial_n u_k(\alpha, \delta)|^2 \leq C \left\{ \int_{\Gamma_R \cup \Gamma_V} |\partial_\tau u_k(\alpha, \delta)|^2 + \int_\Omega |\nabla u_k(\alpha, \delta)|^2 \right\} \tag{42}$$

Using both Lemma 1, the above estimate and the convergence of the sequence $(u_k(\alpha, \delta))_k$ in $H^1(\Omega)$ we are able to prove the convergence of the trace $u_k(\alpha, \delta)|_{\partial\Omega}$ in $H^1(\partial\Omega)$ to zero.

To prove the convergence of the normal derivative, we first begin by identifying its weak limit using adequate raising \mathcal{R} from $H^{1/2}(\partial\Omega)$ into $H^1(\Omega)$ such that we have:

$$\begin{aligned} & \int_{\partial\Omega} \partial_n u_k(\alpha, \delta) v = \int_\Omega \nabla u_k(\alpha, \delta) \nabla (\mathcal{R}(v)) - \lambda_k(\alpha, \delta) \int_\Omega u_k(\alpha, \delta) (\mathcal{R}(v)) \\ & \forall v \in H^{1/2}(\partial\Omega) \end{aligned} \tag{43}$$

This, of course, implies the convergence of $\partial_n u_k(\alpha, \delta)$ in $H^{1/2}(\partial\Omega)$ to a certain f satisfying:

$$\begin{aligned}
\int_{\partial\Omega} f v &= \int_{\Omega} \nabla u_k(D) \nabla(\mathcal{R}(v)) - \lambda_k(D) \int_{\Omega} u_k(D) (\mathcal{R}(v)) \\
&= \int_{\partial\Omega} \partial_n u_k(D) v \\
\forall v &\in H^{1/2}(\partial\Omega)
\end{aligned} \tag{44}$$

From the estimate (42) we deduce that the convergence of the normal derivative is actually weak in $L^2(\partial\Omega)$. The convergence of the norm is then obtained by simple passage to the limit in the first identity of Lemma 2. It is, then, sufficient to identify term by term the result of the passage to the limit obtained with the second identity of Lemma 2 to conclude that convergence of the normal derivative is strong.

Finally, the convergence $H^2(\Omega)$ becomes immediately from the identity of Lemma 1 and the convergence result established above.

5. CONCLUSIONS

In this article, we established the convergence result of eigenvalues and eigenfunctions of Laplacien operator with Wentzell-Robin boundary condition when the Robin parameter tends to infinity.

It may be interesting to be able to solve the problem of convergence to the Neumann problem when Robin parameter tends to zero. It would also be interesting to obtain results of asymptotic behavior both in relation to the setting of the boundary conditions in relation to the variation of geometry.

Another interesting problem would be to examine the convergence of Wentzell control system to the Dirichlet control system (respectively Neumann) when the parameter α tends to infinity (respectively to zero) and to extend the results obtained [9] in the case of Robin-Fourier control.

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